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ON SINGULAR SINGULARLY-PERTURBED INITIAL VALUE PROBLEMS.(U)
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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Sixteen (14) TR-16	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) On Singular Singularly-Perturbed Initial Value Problems,		5. TYPE OF REPORT & PERIOD COVERED (9) Technical Report, ~~May 1977~~
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) R. E. O'Malley, Jr.		8. CONTRACT OR GRANT NUMBER(s) (15) N00014-76-C-0326
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Arizona Department of Mathematics Tucson, Arizona 85721		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR041-466
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematics Branch Office of Naval Research Arlington, Virginia		12. REPORT DATE (11) May 1977
		13. NUMBER OF PAGES ~~13~~ (12) 17p
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release. Distribution unlimited.

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RECEIVED
MAY 26 1977
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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

asymptotic analysis, singular perturbations, bifurcation, initial value problems

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

Unclassified

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On Singular Singularly-Perturbed Initial Value Problems

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DEDICATED TO ARTHUR AND EVA ERDÉLYI ON THE OCCASION OF HIS SEVENTIETH BIRTHDAY

Abstract

Consider the vector initial value problem $\epsilon\dot{y} = f(t,y,\epsilon)$, $y(0) = y^0(\epsilon)$ with $f(t,y,0) = F(t)y + G(t)$ for a singular matrix $F(t)$ of constant rank with stable eigenvalues and zero eigenvalues having simple elementary divisors. This paper shows how to determine the unique limiting solution when the reduced problem $FY_0 + G = 0$ is solvable and how to obtain a uniform asymptotic expansion for the solution on finite t intervals.

* This work was supported in part by the Office of Naval Research under Contract Number N00014-76-C-0326.

1. Introduction

We wish to consider the vector initial value problem

$$\varepsilon\dot{y} = f(t,y,\varepsilon), \quad y(0) = y^0(\varepsilon) \tag{1}$$

in the nearly linear situation that

$$f(t,y,0) \equiv F(t)y + G(t). \tag{2}$$

When $F(t)$ is stable on some interval $0 \leq t \leq T$, it is a classical result of Tihonov [17] that the limiting solution of the initial value problem (1) will converge to the solution $Y_0 = -F^{-1}G$ of the reduced equation

$$0 = F(t)Y_0 + G(t) \tag{3}$$

for $0 < t \leq T$ as the small positive parameter ε tends to zero. (The result continues to be valid for all $t > 0$ if Y_0 decays exponentially to zero as $t \to \infty$ (cf. Hoppensteadt [7]).) The solution generally converges nonuniformly at $t = 0$ since we cannot expect that $y^0(0) = Y_0(0)$. Assuming infinite differentiability of f in t and y and asymptotic series expansions in ε for both f and y^0, we can indeed show that the asymptotic solution $y(t,\varepsilon)$ of (1) is of the form

$$y(t,\varepsilon) = Y(t,\varepsilon) + \Pi(\tau,\varepsilon) \tag{4}$$

throughout $0 \leq t \leq T$ where Y and Π have asymptotic expansions in ε and the terms of Π all tend to zero as the stretched (or boundary layer) variable

$$\tau = t/\varepsilon \tag{5}$$

tends to infinity. Thus, the limiting asymptotic solution for $t > 0$ is provided by the outer solution $Y(t,\varepsilon)$, and the nonuniform convergence of the solution from $y^0(\varepsilon)$ to $Y(0,\varepsilon)$ near $t = 0$ is accomplished through the boundary layer correction $\Pi(\tau,\varepsilon)$.

If $F(t)$ has unstable eigenvalues, we cannot expect the initial value problem to have bounded solutions as $\varepsilon \to 0$ unless we restrict the initial values to a lower dimensional manifold (cf. Levin [11]). It is still reasonable, however, to allow arbitrary initial values if the matrix $F(t)$ is singular. (The usual expansion procedure will then break down.) For an M-vector y, let us assume

(H1) <u>the</u> $M \times M$ <u>matrix</u> $F(t)$ <u>has a constant rank</u> k, $0 \leq k < M$, <u>for all</u> t <u>in</u> $0 < t \leq T$, <u>its nonzero eigenvalues all have negative real parts there, and its null space is spanned by</u> $M - k$ <u>linearly independent eigenvectors</u>.

Under (H1), we shall find that the asymptotic solution of (1) - (2) remains in the form (4) whenever the limiting equation (3) is consistent, i.e. $G \subset \mathcal{R}(F)$, the range of F. Because F is singular, (3) no longer has a unique solution. Its solution is determined up to an arbitrary element of $\mathcal{N}(F)$, the null space of F, so additional analysis is required to determine the unique limiting solution for $t > 0$. This is an instance of a "singular singular-perturbation problem" where the reduced problem (obtained when

$\varepsilon = 0$) has an infinity of solutions in regions of uniform convergence.

More generally, one could consider singular problems (1) where $f_y(t,y,0) = 0$ for some t values along some solutions y of the reduced equation $f(t,y,0) = 0$. Difficult turning point problems would then be faced (cf. Wasow [20]). A somewhat more restricted class of nonlinear problems is considered in O'Malley and Flaherty [15]. Their analysis seeks to eventually develop numerical algorithms for such problems. Among other possible applications are high gain feedback control systems (cf. O'Malley [13]), singular control problems (O'Malley [14]), and certain Cauchy problems in Banach spaces (Gordon [6]). We note that the structure of the asymptotic solutions will generally differ considerably if the elementary divisors of F corresponding to zero eigenvalues are nonlinear. An example is provided by

$$f(t,y,\varepsilon) = F(t,\varepsilon)y \quad \text{for} \quad F = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\varepsilon \\ 0 & -1 & 0 \end{pmatrix}.$$

Further, if F remains nonsingular, but has purely imaginary eigenvalues, rapidly oscillating solutions would result (cf. Hoppensteadt and Miranker [9]).

2. Preliminary Linear Algebra

Under hypothesis (H1), we are guaranteed that the matrix F can be put into its row-reduced form by an orthogonal matrix $E(t)$, i.e.

$$EF = \begin{bmatrix} U \\ 0 \end{bmatrix} \tag{6}$$

where $U(t)$ is a $k \times M$ matrix of rank k. Indeed, E is guaranteed by the singular value decomposition of F, i.e. $F = E'DH$ for orthogonal matrices E and H and diagonal D (cf., e.g., Stewart [16]). E can be obtained numerically by performing a sequence of Householder transformations (cf. Golub [4]). Moreover, the differentiability of E will follow that of the coefficients under the constancy of rank assumption (cf. Golub and Pereyra [5]).

The use of such orthogonal matrices to reduce F to echelon form seems more convenient than the diagonalization procedures which have been more common in the singular perturbation literature. Among other differences, it avoids the use of eigenvectors of F. The success of traditional procedures on closely related problems is demonstrated in Fife [3], Butuzov and Vasil'eva [1], and Vasil'eva [19]. Alternative analogous methods which don't involve orthogonal matrices may be necessary for nonlinear problems (cf. O'Malley [14]).

Writing

$$E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{7}$$

where E_1 is $k \times M$, we have the orthogonality condition

$$E_2F = 0. \tag{8}$$

Furthermore, the orthogonality of E implies that

$$E_1E_2' = 0, \quad E_1E_1' = I_k, E_2E_2' = I_{M-k}, \text{ and } E_1'E_1 + E_2'E_2 = I_M. \tag{9}$$

These last relationships imply that

$$P = E_1'E_1 = P^2 \text{ and } Q = E_2'E_2 = Q^2 \tag{10}$$

are complementary projections. Indeed, a direct sum decomposition of M-space results with

$$\mathcal{R}(Q) = \mathcal{N}(F') \quad \text{and} \quad \mathcal{R}(P) = \mathcal{R}(F). \tag{11}$$

This follows since $F'Q = (E_2F)'E_2 = 0$ shows that Q projects into $\mathcal{N}(F')$ and the dimensions of $\mathcal{R}(Q)$ and $\mathcal{N}(F')$ are both $M - k$. Finally, (7) and (8) show that $EFE' = \begin{pmatrix} E_1FE_1' & E_1FE_2' \\ 0 & 0 \end{pmatrix}$ where the $k \times k$ matrix

$$S \equiv E_1FE_1' \text{ is stable} \tag{12}$$

since it shares the k stable eigenvalues of F. Indeed, the purpose of (H1) is to provide an orthogonal matrix E such that EF is row-reduced with EFE' having k stable eigenvalues.

3. The Transformation Approach

Making the one-one transformation

$$z = E(t)y, \tag{13}$$

z will satisfy

$$\varepsilon z' = EFE'z + EG + \varepsilon[\dot{E}E'z + \frac{1}{\varepsilon} E(f(t,E'z,\varepsilon) - f(t,E'z,0))]. \tag{14}$$

Setting

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} E_1y \\ E_2y \end{pmatrix}, \tag{15}$$

then, provides a transformed singular perturbation problem

$$(16)\qquad \begin{cases} \varepsilon z_1' = Sz_1 + E_1FE_2'z_2 + E_1G + \varepsilon g_1(z_1,z_2,t,\varepsilon) \\ \varepsilon z_2' = E_2G + \varepsilon g_2(z_1,z_2,t,\varepsilon) \end{cases}$$

where $g_2 = \dot{E}_2E'z + \frac{1}{\varepsilon}E_2(f(t,E'z,\varepsilon) - f(t,E'z,0))$. The asymptotic behavior of solutions to (16) differs considerably in the two cases $E_2G = 0$ and $E_2G \neq 0$. Since $E_2G = 0$ if and only if $QG = 0$, Q projecting onto $N(F')$, we're guaranteed that $E_2G = 0$ if and only if the reduced problem (3) is consistent. Taking

$$(17)\qquad E_2G \equiv 0 \quad \text{throughout} \quad [0,T],$$

then, reduces (16) to a "nonsingular singular-perturbation problem." For it, classical singular perturbation theory (cf., e.g., Hoppensteadt [8] or O'Malley [12]) implies that (16) - (17) has a limiting solution for $t > 0$ which is the unique solution $\begin{pmatrix} Z_{10} \\ Z_{20} \end{pmatrix}$ of the reduced problem

$$(18)\qquad \begin{cases} 0 = SZ_{10} + E_1FE_2'Z_{20} + E_1G \\ \dot{Z}_{20} = g_2(Z_{10},Z_{20},t,0), \qquad Z_{20}(0) = E_2(0)y^0(0), \end{cases}$$

presuming that the corresponding nonlinear initial value problem

$$(19)\qquad \dot{Z}_{20} = g_2(-S^{-1}E_1(FE_2'Z_{20} + G),Z_{20},t,0), \quad Z_{20}(0) = E_2(0)y^0(0)$$

has a solution for $0 \leq t \leq T$. Furthermore, that theory also shows that the nonuniform convergence of z near $t = 0$ is determined as the decaying solution of the boundary layer problem

$$(20) \quad \begin{cases} \dfrac{dT_{10}}{d\tau} = S(0)T_{10}, \quad T_{10}(0) = E_1(0)y^0(0) - Z_1(0) \\ \dfrac{dT_{20}}{d\tau} = 0. \end{cases}$$

Thus, $T_{10}(\tau) = e^{-S(0)\tau}\, T_{10}(0)$ and $T_{20}(0) = 0$ for $\tau \geq 0$. The bifurcation character (cf., e.g., Cesari [2]) of these limiting problems makes them differ from the classical ones. Here, the reduced problem (18) consists of an algebraic problem for a k-vector and an initial value problem for an $M - k$ vector, while the boundary layer problem (20) involves a k dimensional initial value problem and an $M - k$ dimensional algebraic problem.

4. A Direct Solution

Since the boundary layer correction $\Pi(\tau,\varepsilon)$ will be asymptotically negligible for each $t > 0$, the outer solution

$$(21) \qquad Y(t,\varepsilon) \sim \sum_{j=0}^{\infty} Y_j(t)\varepsilon^j$$

must satisfy the original nearly linear system (1) as a power series in ε. Rewriting the equation as

$$(22) \qquad F(t)Y = -G(t) - [f(t,Y,\varepsilon) - f(t,Y,0)] + \varepsilon\dot{Y}$$

and substituting in the expansion (21) implies that the terms of (21) must successively satisfy the algebraic equations

$$F(t)Y_j = \xi_{j-1}(t) \tag{23}$$

where

$$\begin{cases} \xi_{-1} = -G, \quad \xi_0 = \dot{Y}_0 - f_\varepsilon(t,Y_0,0), \\ \text{and} \\ \xi_k = \dot{Y}_k - f_{y\varepsilon}(t,Y_0,0)Y_k + \eta_{k-1}(t), \quad k \geq 1 \end{cases} \tag{24}$$

for a known function η_{k-1} of $Y_0, Y_1, \ldots, Y_{k-1}$. Assuming consistency of (3), $G \subset R(F)$ so, by (8), $QG = 0$. Consistency of (23), by the Fredholm alternative, requires $Q\xi_{j-1} = 0$, $j \geq 1$, so we obtain the singular differential equations

$$\begin{cases} Q\dot{Y}_0 = Qf_\varepsilon(t,Y_0,0) \\ \text{and} \\ Q\dot{Y}_k = Qf_{y\varepsilon}(t,Y_0,0)Y_k - Q\eta_{k-1}, \quad k \geq 1. \end{cases} \tag{25}$$

Taking (23) and (25) together nearly allows us to obtain the outer solution Y termwise. Specifically, multiplying (23) by E_1 implies that $E_1F(E_1'E_1 + Q)Y_j = E_1\xi_{j-1}$, $j \geq 0$, so the invertibility of $S = E_1FE_1'$ and the definition of P imply that

$$PY_j = -AF(QY_j) + A\xi_{j-1} \tag{26}$$

where $A = E_1'S^{-1}E_1$. Thus PY_j and thereby Y_j are completely determined as linear functions of QY_j. Putting (25) and (26) together, then, implies the differential equations

$$(27) \quad \begin{cases} (QY_0)^{\cdot} = \dot{Q}(\mathcal{B}(QY_0) - \mathcal{A}G) + Qf_{\varepsilon}(t,\mathcal{B}(QY_0) - \mathcal{A}G,0) \\ \text{and} \\ (QY_k)^{\cdot} = [\dot{Q} + Qf_{y\varepsilon}(t,Y_0,0)][\mathcal{B}(QY_k) + \mathcal{A}\xi_{k-1}] - Q\eta_{k-1}, \quad k \geq 1 \end{cases}$$

since (26) implies that

$$(28) \qquad Y_j = \mathcal{B}(QY_j) + \mathcal{A}\xi_{j-1}$$

for the projection $\mathcal{B} = I - \mathcal{A}F$. Note that the representation (4) implies the yet unspecified initial values

$$(29) \qquad Q(0)Y_k(0) = Q(0)y_k^0 - Q(0)\Pi_k(0), \quad k \geq 0$$

needed to completely determine the outer solution $Y(t,\varepsilon)$.

If we anticipate that $Q(0)\Pi_0(0) = 0$ (paralleling the result that $T_{20}(0) = 0$ in (20)), it will follow that the limiting outer solution is given by

$$(30) \qquad Y_0 = \mathcal{B}(QY_0) - \mathcal{A}G$$

where QY_0, the projection of the limiting solution onto $\mathcal{N}(F')$, satisfies the nonlinear initial value problem

$$(31)\quad \begin{cases} (QY_0)^{\bullet} = \dot{Q}(\mathcal{B}(QY_0) - \mathcal{A}G) + Qf_\varepsilon(t,\mathcal{B}(QY_0) - \mathcal{A}G,0) \\ \\ Q(0)Y_0(0) = Q(0)y^0(0). \end{cases}$$

We shall assume that the solution to (31) exists throughout $0 \leq t \leq T$ (in analogy to our existence assumption for (19)). We note that (31) shows that the limiting solution of (1) - (2) is determined by a dynamical system in an $M - k$ dimensional space. Further, we observe that unique solutions QY_k and Y_k for $k > 0$ follow from the linear differential equations of (27) and the algebraic relation (28).

By the form of (4), the boundary layer correction Π must satisfy the system

$$(32)\quad \frac{d\Pi}{d\tau} = \varepsilon\frac{dy}{dt} - \varepsilon\frac{dY}{dt} = f(\varepsilon\tau,Y(\varepsilon\tau,\varepsilon) + \Pi(\tau,\varepsilon),\varepsilon) - f(\varepsilon\tau,Y(\varepsilon\tau,\varepsilon),\varepsilon)$$

and the terms of its expansion

$$(33)\quad \Pi(\tau,\varepsilon) \sim \sum_{j=0}^{\infty} \Pi_j(\tau)\varepsilon^j$$

should tend to zero as τ tends to infinity. In the quasilinear situation when (2) holds, equating coefficients successively in (32) implies the linear system

$$(34)\quad \frac{d\Pi_j}{d\tau} = F(0)\Pi_j + \zeta_{j-1}(\tau)$$

where ζ_{j-1} is a linear combination of the preceding coefficients Π_ℓ, $\ell < j$, with polynomial coefficients in τ, and $\zeta_{-1} \equiv 0$. Since $QF = 0$, it follows that $\frac{d}{d\tau}(Q(0)\Pi_j(\tau)) = Q(0)\zeta_{j-1}(\tau)$. Thus, the decay requirement implies that

$$Q(0)\Pi_j(\tau) = -Q(0)\int_\tau^\infty \zeta_{j-1}(s)ds. \tag{35}$$

We note, in particular, that (35) allows the termwise determination of the initial values (29) needed to fix the outer solution. It remains to successively find the $P(0)\Pi_j(\tau)$'s. Multiplying (34) by $E_1(0)$ implies that $\frac{d}{d\tau}(E_1(0)\Pi_j) = S(0)(E_1(0)\Pi_j) + E_1(0)F(0)(Q(0)\Pi_j) + E_1(0)\zeta_{j-1}$. Using (35) and $I_M = E_1'(0)E_1(0) + Q(0)$ implies the decaying solutions

$$\begin{aligned}\Pi_j(\tau) &= E_1'(0)e^{S(0)\tau}E_1(0)\Pi_j(0) - Q(0)\int_\tau^\infty \zeta_{j-1}(s)ds \\ &\quad + E_1'(0)\int_0^\tau e^{S(0)(\tau-r)}E_1(0)[\zeta_{j-1}(r) - F(0)Q(0)\int_r^\infty \zeta_{j-1}(s)ds]dr.\end{aligned} \tag{36}$$

Thus, the boundary layer terms are determined successively as exponentially decaying vectors up to the initial value $E_1(0)\Pi(0,\varepsilon)$, just as the outer expansion coefficients are determined up to $Q(0)Y(0,\varepsilon)$. Since the limiting outer solution is known from (30) - (31) and higher order terms follow successively from (27) - (29), the unique higher order boundary layer correction terms follow from (36) with

$$E_1(0)\Pi_j(0) = E_1(0)(y_j^0 - Y_j(0)), \quad j \geq 0. \tag{37}$$

For $j = 0$, we'll have

$$\Pi_0(\tau) = E_1'(0)e^{S(0)\tau}E_1(0)(y^0(0) - B(0)Q(0)y^0(0) - A(0)G(0)). \tag{38}$$

Since $QE_1' = 0$, it follows that Π_0 lies within the k dimensional

$\mathcal{R}(F) = \mathcal{R}(P)$. The function of this zero order boundary layer correction term, then, is to "instantly" transfer from the prescribed initial vector $y^0(0)$ to $Y_0(0^+)$ where the reduced equation (3) is satisfied. Finally, we observe that the expansion (4) which we've formally generated will, of course, agree with that obtained by our earlier transformation procedure and is justified by it.

Summarizing, we have

<u>Theorem</u>

<u>Under hypothesis</u> (H1); (H2), <u>that</u> (3) <u>is consistent</u>; <u>and</u> (H3), <u>that the solution</u> (QY_0) <u>of</u> (34) <u>exists for</u> $0 \le t \le T$, <u>the unique solution of the initial value problem</u> (1) - (2) <u>is asymptotically of the form</u> (4) <u>uniformly in</u> $0 \le t \le T$ <u>as</u> $\varepsilon \to 0$.

5. The Inconsistent Problem

If $E_2G \neq 0$, the transformed system (16) cannot be easily solved as (16) - (17) was. Then, the limiting system (3) has no solutions and we cannot expect to have an outer solution of the form (21).

By variation of parameters, we know that solution of (1) - (2) must satisfy the integral equation

$$(39)\quad y(t,\varepsilon) = \mathcal{Y}(t,\varepsilon)y^0(\varepsilon) + \frac{1}{\varepsilon}\int_0^t \mathcal{Y}(t,\varepsilon)\mathcal{Y}^{-1}(s,\varepsilon)\{G(t) + f(t,y,\varepsilon) - f(t,y,0)\}dt$$

where $\mathcal{Y}$ is the fundamental matrix solution of the linear system

$$(40)\qquad \dot{\mathcal{Y}} = \frac{1}{\varepsilon}F(t)\mathcal{Y}, \quad \mathcal{Y}(0,\varepsilon) = I_M.$$

Such fundamental matrices can be constructed following the work of Turrittin and earlier authors (cf. Turrittin [18]). Under natural assumptions, we can show that $||\mathcal{Y}(t,\varepsilon)\mathcal{Y}^{-1}(s,\varepsilon)|| \leq K$ for $0 \leq s,t \leq T$ for some constant K provided no eigenvalues of F are unstable. When F is stable, we instead have a decaying bound $Ke^{-\kappa(t-s)/\varepsilon}$ and this allows the corresponding outer solution for (1) - (2) to be bounded. When F has eigenvalues with zero real parts, other estimates are appropriate (cf. Hoppensteadt and Miranker [9]).

For

$$f(t,y,\varepsilon) = F(t,\varepsilon)y + G(t,\varepsilon) \tag{41}$$

linear in y, (39) makes it natural to seek the solution to (1) in the form

$$y(t,\varepsilon) = \frac{1}{\varepsilon} Z(t,\varepsilon) + \Pi(\tau,\varepsilon) \tag{42}$$

when hypothesis (H1) holds for F_0 with $(F(t,\varepsilon),G(t,\varepsilon)) \sim \sum_{j=0}^{\infty} (F_j(t),G_j(t))\varepsilon^j$.

Here the outer solution $\frac{1}{\varepsilon} Z(t,\varepsilon)$ and the boundary layer correction Π have power series expansions in ε and $\Pi \to 0$ as $\tau \to \infty$. Now the limiting solution $\frac{1}{\varepsilon} Z_0(t)$ for $t > 0$ satisfies the homogeneous system $F_0(t)Z_0 = 0$. Applying the Fredholm alternative to the equation for F_0Z_1 requires that $Q(\dot{Z}_0 - F_1Z_0 - G_0) = 0$ where Q is the projection onto $N(F_0')$. Using our earlier notation, then, we have the unbounded limiting solution $\frac{1}{\varepsilon} Z_0(t)$ with

$$Z_0 = \mathcal{B}(QZ_0) \tag{43}$$

where QZ_0 satisfies the linear system

$$(QZ_0)^{\bullet} = (\dot{Q} + QF_1)\mathcal{B}(QZ_0) + QG_0, \quad Q(0)Z_0(0) = 0. \tag{44}$$

The boundary layer correction is calculated as before. When $QG_0 \equiv 0$, $Z_0 = 0$ and the solution reduces to the form (4) obtained for the consistent problem.

If f is nonlinear in y, the expansion (42) cannot generally be used. (A term like y^k would provide a term $\frac{1}{\varepsilon^k} Z_0^k$ in the outer region.) For polynomial dependence on y, an appropriate scheme might be devised for the inconsistent problem though we shall not do so (cf. Kersten [10], however, for a related discussion concerning special boundary value problems).

Acknowledgment

This paper is based on earlier work with Joseph Flaherty of Rensselaer Polytechnic Institute and owes much to interaction with him. The author also benefitted from conversations with Wiktor Eckhaus of Rijksuniversiteit Utrecht.

References

1. V. F. Butuzov and A. B. Vasil'eva, "Differential and difference equation systems with a small parameter for the case in which the unperturbed (singular) system is in the spectrum." Soviet Math. Dokl. 6 (1970), 499-510.

2. L. Cesari, "Alternative methods in nonlinear analysis," International Conference on Differential Equations, H. A. Antosiewicz, editor, Academic Press, New York, 1975, 95-148.

3. P. C. Fife, "Singular perturbation problems whose degenerate forms have many solutions," Applicable Analysis 1 (1972), 331-358.

4. G. H. Golub, "Numerical methods for solving linear least squares problems," Numer. Math. 7 (1965), 206-216.

5. G. H. Golub and V. Pereyra, "Differentiation of pseudoinverses, separable nonlinear least square problems, and other tales," Generalized Inverses and Applications, M. Z. Nashed, editor, Academic Press, New York, 1976, 303-324.

6. N. Gordon, "Matched asymptotic expansion solutions of nonlinear partial differential equations with a small parameter," SIAM J. Math. Anal. 6 (1975), 1007-1016.

7. F. C. Hoppensteadt, "Singular perturbations on the infinite interval," Trans. Amer. Math. Soc. 123 (1966), 521-535.

8. F. Hoppensteadt, "Properties of solutions of ordinary differential equations with a small parameter," Comm. Pure Appl. Math. 24 (1971), 807-840.

9. F. C. Hoppensteadt and W. L. Miranker, "Differential equations having rapidly changing solutions: Analytic methods for weakly nonlinear systems," J. Differential Equations 22 (1976), 237-249.

10. P. H. M. Kersten, "Uniform asymptotic expansions for a class of nonlinear systems of differential equations containing a small parameter," report, Twente University of Technology, March 1972.

11. J. J. Levin, "The asymptotic behavior of the stable initial manifolds of a system of nonlinear differential equations," Trans. Amer. Math. Soc. 85 (1957), 357-368.

12. R. E. O'Malley, Jr., Introduction to Singular Perturbations, Academic Press, New York, 1974.

13. R. E. O'Malley, Jr., "High gain feedback systems as singular singular-perturbation problems," Proceedings, 1977 Joint Automatic Control Conference.

14. R. E. O'Malley, Jr., "Partially singular control problems as singular singular-perturbation problems," to appear.

15. R. E. O'Malley, Jr. and J. E. Flaherty, "Singular singular-perturbation problems," Lecture Notes in Math.

16. G. W. Stewart, Introduction to Matrix Computations, Academic Press, New York, 1973.

17. A. N. Tihonov, "Systems of differential equations containing small parameters in the derivatives," Mat. Sb. 31 (73) (1952), 575-586.

18. H. L. Turrittin, "Asymptotic expansions of solutions of systems of ordinary differential equations containing a parameter," Contrib. Theory Nonlinear Oscillations 2 (1952), 81-116.

19. A. B. Vasil'eva, "Singularly perturbed systems containing indeterminacy in the case of degeneracy," Soviet Math. Dokl. 16 (1975), 1121-1125.

20. W. Wasow, Asymptotic Expansions for Ordinary Differential Equations, Wiley, New York, 1965.